(b) What is the probability to win? (i.e to have at least three matching numbers)

*In order to have a win, we need to have exactly 3, 4 or 5 matching numbers. We already know the probabilities for exactly 3 or 5 matching numbers. What remains, is the probability for exactly 4 matching numbers.*

*If we use the above formula and substitute the 3 by a 4, we get*

$$P(\text{ ``}4 \text{ matching numbers''} ) = \frac{\binom{5}{4}\binom{49-5}{5-4}}{\binom{49}{5}} = \frac{5 \cdot 49}{\binom{49}{5}} \approx 0.000128$$

*In total the probability to win is:*

$$\begin{aligned} P(\text{ ``win''} ) &= P(\text{ ``}3 \text{ matching numbers''} ) + P(\text{ ``}4 \text{ matches''} ) + P(\text{ ``}5 \text{ matches''} ) = \\ &= \frac{1 + 5 \cdot 49 + 231}{1906884} = 477 : 1906884 \approx 0.00025. \end{aligned}$$

Please note: In the previous examples we've used parentheses ( ), see definition , to indicate that the order of the elements inside matters. These constructs are called *tuples*.
If the order of the elements does not matter, we use { } - the usual symbol for *sets*.

## 1.5 Conditional Probabilities

**Example 1.5.1**
A box contains 4 computer chips, two of them are defective.
*Obviously, the probability to draw a defective chip in one random draw is 2/4 = 0.5.*
We analyze this chip now and find out that it is a good one.
If we draw now, what is the probability to draw a defective chip?
*Now, the probability to draw a defective chip has changed to 2/3.*

Conclusion: The probability of an event $A$ may change if we know (before we start the experiment for $A$) the outcome of another event $B$.
We need to add another term to our mathematical description of probabilities:

| **Real World** | **Mathematical World** |
|---|---|
| assessment of "chance" given additional, partial information | conditional probability of one event $A$ given another event $B$.<br>write: $P(A\|B)$ |

**Definition 1.5.1 (conditional probability)**
The conditional probability of event $A$ given event $B$ is defined as:

$$P(A|B) := \frac{P(A \cap B)}{P(B)} \qquad \text{if } P(B) \neq 0.$$

**Example 1.5.2**
A lot of unmarked Pentium III chips in a box is as

| | 400 mHz | 500 mHz | |
|---|---|---|---|
| Good | 480 | 490 | 970 |
| Defective | 20 | 10 | 30 |
| | 500 | 500 | 1000 |

Drawing a chip at random has the following probabilities:

| | | |
|---|---|---|
| $P(D) = 0.03$ | $P(G) = 0.97$ | check: these two must sum to 1. |
| $P(400mHz) = 0.5$ | $P(500mHz) = 0.5)$ | check: these two must sum to 1, too. |

$P(D \text{ and } 400mHz) = 20/1000 = 0.02$
$P(D \text{ and } 500mHz) = 10/1000 = 0.01$

Suppose now, that I have the partial information that the chip selected is a 400 mHz chip.
What is now the probability that it is defective?
*Using the above formula, we get*

$$P(\text{ chip is } D| \text{ chip is } 400mHz) = \frac{P(\text{ chip is } D \text{ and chip is } 400mHz)}{P(\text{ chip is } 400mHz)} = \frac{0.02}{0.5} = 0.04.$$

*i.e. knowing the speed of the chip influences our probability assignment to whether the chip is defective or not.*

Note: Rewriting the above definition of conditional probability gives:

$$P(A \cap B) = P(B) \cdot P(A|B), \tag{1.1}$$

i.e. knowing two out of the three probabilities gives us the third for free.
We have seen that the occurrence of an event $B$ may change the probability for an event $A$. If an event $B$ does **not** have any influence on the probability of $A$ we say, that the events $A$ and $B$ are **independent**:

## 1.6 Independence of Events

**Definition 1.6.1 (Independence of Events)**
Two events $A$ and $B$ are called *independent*, if

$$P(A \cap B) = P(A) \cdot P(B)$$

(Alternate definition: $P(A|B) = P(A)$)

Independence is the mathematical counterpart of the everyday notion of “unrelatedness” of two events.

**Example 1.6.1** *Safety System at a nuclear reactor*
Suppose there are two physically separate safety systems $A$ and $B$ in a nuclear reactor. An “incident” can occur only when both of them fail in the event of a problem.
Suppose the probabilities for the systems to fail in a problem are:

$$P(A \text{ fails}) = 10^{-4} \qquad P(B \text{ fails}) = 10^{-8}$$

*The probability for an incident is then*

$$\begin{aligned} P(\text{ incident }) &= P(A \text{ and } B \text{ fail at the same time } = \\ &= P(A \text{ fails and } B \text{ fails}) \end{aligned}$$

*Using that A and B are independent from each other, we can compute the intersection of the events that both systems fail as the product of the probabilities for individual failures:*

$$P(A \text{ fails and } B \text{ fails}) \overset{A,B \text{ independent}}{=} P(A \text{ fails }) \cdot P(B \text{ fails})$$

*Therefore the probability for an incident is:*

$$P(\text{ incident }) = P(A \text{ fails }) \cdot P(B \text{ fails}) = 10^{-4} \cdot 10^{-8} = 10^{-12}.$$

**Comments** The safety system at a nuclear reactor is an example for a “parallel system”
A parallel system consists of $k$ components $c_1, \ldots, c_k$, that are arranged as drawn in the diagram **??**.

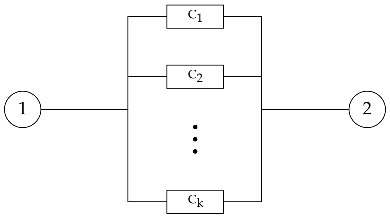


Figure 1.1: *Parallel system with k components.*

The system works as long as there is at least one unbroken path between 1 and 2 (= at least one of the components still works).
Under the assumption that all components work independently from each other, it is fairly easy to compute the probability that a parallel system will fail:

$$
\begin{aligned}
P(\text{ system fails }) &= P(\text{ all components fail }) = \\
&= P(c_1 \text{ fails} \cap c_2 \text{ fails} \cap \ldots c_k \text{ fails}) \overset{\text{components are independent}}{=} \\
&= P(c_1 \text{ fails}) \cdot P(c_2 \text{ fails}) \cdot \ldots \cdot P(c_k \text{ fails})
\end{aligned}
$$

A similar kind of calculation can be done for a “series system”. A series system, again, consists of $k$ supposedly independent components $c_1, \ldots, c_k$ arranged as shown in diagram **??**.

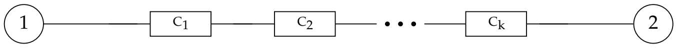


Figure 1.2: *Series system with k components.*

This time, the system only works, if all of its components are working.
Therefore, we can compute the probability that a series system works as:

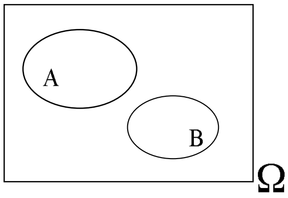
$$
\begin{aligned}
P(\text{ system works }) &= P(\text{ all components work }) = \\
&= P(c_1 \text{ works} \cap c_2 \text{ works} \cap \ldots c_k \text{ works}) \overset{\text{components are independent}}{=} \\
&= P(c_1 \text{ works}) \cdot P(c_2 \text{ works}) \cdot \ldots \cdot P(c_k \text{ works})
\end{aligned}
$$

Please note that based on the above probabilities it is easy to compute the probability that a parallel system is working and a series system fails, respectively, as:

$$
\begin{aligned}
P(\text{ parallel system works }) &\overset{Thm 1.3.1}{=} 1 - P(\text{ parallel system fails}) \\
P(\text{ series system fails }) &\overset{Thm 1.3.1}{=} 1 - P(\text{ parallel system works})
\end{aligned}
$$

The probability that a system works is sometimes called the system’s *reliability.* Note that a parallel system is very reliable, a series system usually is very unreliable.
Warning: independence and disjointness are two very different concepts!

Disjointness:

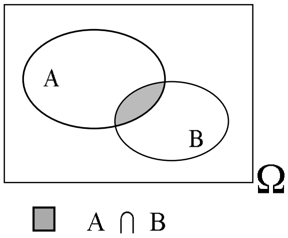


A, B are disjoint

If $A$ and $B$ are disjoint, their intersection is empty, has therefore probability 0:

$$P(A \cap B) = P(\emptyset) = 0.$$

Independence:



If $A$ and $B$ are independent events, the probability of their intersection can be computed as the product of their individual probabilities:

$$P(A \cap B) = P(A) \cdot P(B)$$

If neither of $A$ or $B$ are empty, the probability for the intersection will not be 0 either!

The concept of independence between events can be extended to more than two events:

**Definition 1.6.2** **(Mutual Independence)**
A list of events $A_1, \ldots, A_n$ is called *mutually independent*, if for any subset $\{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$ of indices we have:

$$P(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}) = P(A_{i_1}) \cdot P(A_{i_2}) \cdot \ldots \cdot P(A_{i_k}).$$

Note: for more than 3 events pairwise independence does not imply mutual independence.